1. Lines in $\mathbb{R}^2$
2. Section 12.5: lines and planes in space
3. Application: perspective projection
Convection

- As from now on we will identify points with terminal points of vectors in standard position:

\[ P = \mathbf{v} \]

- We will abandon the notation \( \langle x_1, \ldots, x_n \rangle \) and use \((x_1, \ldots, x_n)\) instead.
Definition

A line in $\mathbb{R}^2$ is defined by an equation of the form

\[ \ell: ax + by = c \]  

(*)

with $a$, $b$ and $c$ real numbers.
Definition

A line in \( \mathbb{R}^2 \) is defined by an equation of the form

\[
\ell: ax + by = c
\]  

(*)

with \( a, b \) and \( c \) real numbers.

The line \( \ell \) consists of the points that satisfy equation (\*):

\[
\ell = \{(x, y) \mid ax + by = c\}.
\]
**Definition**

A line in $\mathbb{R}^2$ is defined by an equation of the form

$$\ell: ax + by = c$$

(*)

with $a$, $b$ and $c$ real numbers.

- The line $\ell$ consists of the points that satisfy equation (*):
  $$\ell = \{(x, y) | ax + by = c\}.$$  
- The line $\ell$ is the **solution set** of equation (*).
**Definition**

A **parametrisation** of the line \( \ell \) is a function \( r: \mathbb{R} \to \mathbb{R}^2 \) such that \( r(t) \) reaches all points of \( \ell \) while \( t \) runs through all real numbers.

![Diagram of a line and a point](image.png)

\( \ell \)

\( r(t) \)

\( y \)

\( x \)
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- The number \( t \) is called the parameter.
**Definition**

A *parametrisation* of the line $\ell$ is a function $\mathbf{r}: \mathbb{R} \to \mathbb{R}^2$ such that $\mathbf{r}(t)$ reaches all points of $\ell$ while $t$ runs through all real numbers.

- The number $t$ is called the **parameter**.
- The line $\ell$ is the set of all points $\mathbf{r}(t)$:
  $$\ell = \{ \mathbf{r}(t) \mid t \in \mathbb{R} \}.$$
Definition

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- The function $r(t)$ has two components that both depend on $t$:
  \[ r(t) = (x(t), y(t)) . \]
A parametrisation of the line \( \ell \) is a function \( r: \mathbb{R} \rightarrow \mathbb{R}^2 \) such that \( r(t) \) reaches all points of \( \ell \) while \( t \) runs through all real numbers.

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- The function \( r(t) \) has two components that both depend on \( t \):
  \[
  r(t) = (x(t), y(t)).
  \]
- Functions like \( r \) with values in \( \mathbb{R}^n \) are called **vector functions**.
Example

Given is the line $\ell: 2x + 3y = 6$. Find a parametrisation of $\ell$. 

Choose $x$ as parameter: $t = x$.

Solve $y$ from the equation $2t + 3y = 6$:

$$y = \frac{6 - 2t}{3} = 2 - \frac{2}{3}t.$$ 

A parametrisation of $\ell$ is $\ell: r(t) = (t, 2 - \frac{2}{3}t), t \in \mathbb{R}$. 

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**UNIVERSITY OF TWENTE.**

**Introduction to Mathematics and Modeling**

**Lecture 6: Points, lines and planes**
Example

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  \]
- \[
  \begin{array}{cccc}
  t & x(t) & y(t) & \mathbf{r}(t) \\
  0 & 0 & 2 & (0, 2)
  \end{array}
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- \[
  \begin{array}{cccc}
  t & x(t) & y(t) & \mathbf{r}(t) \\
  0 & 0 & 2 & (0, 2) \\
  1.5 & 1.5 & 1 & (1.5, 1) \\
  \end{array}
  \]
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  \]

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x(t)$</th>
<th>$y(t)$</th>
<th>$\mathbf{r}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>1</td>
<td>(1.5, 1)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>(3, 0)</td>
</tr>
</tbody>
</table>
Example

Find an equation for the line

\[ \ell: (3t, 2 - 2t), \quad t \in \mathbb{R}. \]
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- The parametric equations are

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  x &= 3t, \\
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\end{align*}
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- From the second parametric equation follows
  \[
  y = 2 - \frac{2}{3}x,
  \]
Example

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  $$y = 2 - \frac{2}{3}x,$$
  $$3y = 6 - 2x,$$
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- **The parametric equations** are
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- **Eliminate** \( t \): from the first parametric equation follows
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- **From the second parametric equation follows**
  \[
  y = 2 - \frac{2}{3}x, \\
  3y = 6 - 2x, \\
  2x + 3y = 6.
  \]
Theorem

For every line $\ell$ there exist numbers $p_1$, $p_2$, $v_1$ and $v_2$ such that

\[ r(t) = (p_1 + v_1 t, p_2 + v_2 t) \quad t \in \mathbb{R}. \]
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$$r(t) = (p_1 + v_1 t, p_2 + v_2 t) \quad t \in \mathbb{R}.$$ 

Write $r(t)$ as follows:

$$r(t) = (p_1, p_2) + t(v_1, v_2).$$
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For every line \( \ell \) there exist numbers \( p_1, p_2, v_1 \) and \( v_2 \) such that

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- Write \( \mathbf{r}(t) \) as follows:
  \[
  \mathbf{r}(t) = (p_1, p_2) + t(v_1, v_2).
  \]
- The vector \( \mathbf{p} = (p_1, p_2) \) is called a **support vector** of \( \ell \).
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- The vector \( \mathbf{v} = (v_1, v_2) \) is called a direction vector of \( \ell \).
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- Define $q = r(1)$, then
  $$r(1) = p + v, \quad \text{dus} \quad v = q - p.$$
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- Define $q = r(1)$, then
  $$r(1) = p + v, \quad \text{dus} \quad v = q - p.$$  
- The parametrised vector form of $\ell$ is
  $$\ell: r(t) = p + tv \quad t \in \mathbb{R}.$$
Example

Find a support- and a direction vector of the line $\ell: 2x + 3y = 6$, and find a parametrised vector form of $\ell$. 

![Graph of the line $\ell$ with support- and direction vectors.](image-url)
Example

Find a support- and a direction vector of the line $\ell : 2x + 3y = 6$, and find a parametrised vector form of $\ell$.

A parametrisation of $\ell$ is

$$\ell : \mathbf{r}(t) = (t, 2 - \frac{2}{3}t), \ t \in \mathbb{R}.$$
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  \[
  \mathbf{r}(t) = (0, 2) + t(1, -\frac{2}{3}).
  \]
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\]

Choose a support- and a direction vector
\[
\mathbf{p} = (0, 2) \quad \text{and} \quad \mathbf{v} = (1, -\frac{2}{3})
\]
Example

Find a parametrisation and an equation of the line $\ell$ that passes through the points $P = (-1, -1)$ and $Q = (1, 3)$.
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Example

Find a parametrisation and an equation of the line \( \ell \) that passes through the points \( P = (-1, -1) \) and \( Q = (1, 3) \).

- Define \( p = (-1, -1) \) and \( q = (1, 3) \).
- Define \( v = q - p = (2, 4) \), then a parametrisation of \( \ell \) is
  \[
  \ell : r(t) = p + tv = (-1, -1) + t(2, 4) \\
  = (2t - 1, 4t - 1).
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- The parametric equations are
  \[
  \begin{cases}
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  y = 4t - 1,
  \end{cases}
  \]
  hence $t = \frac{x + 1}{2}$. 
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- The parametric equations are
  $$\begin{cases} x = 2t - 1, \\ y = 4t - 1, \end{cases}$$
  hence $t = \frac{x+1}{2}$.
- Substitute this in the second equation:
  $$y = 2(x + 1) - 1 = 2x + 1 \quad \text{or} \quad y - 2x = 1$$
1. Let \( P = (2, 1) \) and \( Q = (-1, -1) \). The line passing through \( P \) and \( Q \) is called \( \ell \).

   (i) Find an equation of \( \ell \).
   (ii) Find a direction vector and a support vector of \( \ell \).

2. The line \( \ell \) is defined by the equation \( y = 2x + 1 \). Find a parametrised vector form for \( \ell \).

3. The line \( \ell \) has parametrisation \( \mathbf{x}(t) = \left( 1 - \frac{1}{2} t, t - 1 \right) \) with \( t \in \mathbb{R} \).

   (i) Find an equation of \( \ell \).
   (ii) Find the intersection of \( \ell \) and the line \( m \) defined by the parametrised vector form \( (-2, 2) + t(1, 1) \).

4. The line \( \ell \) has support vector \((3, 2)\) and direction vector \((2, -1)\). The line \( m \) has support vector \((-2, -3)\) and direction vector \((1, 2)\). Find the intersection point of \( \ell \) and \( m \).
Definition

Let \( p \) and \( v \neq 0 \) be vectors. The **parametrised vector form** of the line through \( p \) and parallel to \( v \) is

\[
r(t) = p + tv, \quad t \in \mathbb{R}.
\]
Definition

Let \( \mathbf{p} \) and \( \mathbf{v} \neq 0 \) be vectors. The **parametrised vector form** of the line through \( \mathbf{p} \) and parallel to \( \mathbf{v} \) is

\[
\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbb{R}.
\]

- The vector \( \mathbf{p} \) is called a **support vector** and the vector \( \mathbf{v} \) is called a **direction vector** of the line.
**Definition**

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- The vector \( p \) is called a **support vector** and the vector \( v \) is called a **direction vector** of the line.
- If \( r(t) = (f(t), g(t), h(t)) \), then the equations

\[
\begin{align*}
x &= f(t), \\
y &= g(t), \\
z &= h(t)
\end{align*}
\]

are called the **parametric equations** of the line.
Example 1

Find the parametric equations of the line $\ell$ through $(-2, 0, 4)$ in the direction $v = 2i + 4j - 2k$

$= (2, 4, -2)$. 
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- Define $p = P_0 = (-2, 0, 4)$. 
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- Define $p = P_0 = (-2, 0, 4)$.
- A parametrisation of $\ell$ is
  $$\ell: \mathbf{r}(t) = p + tv = (-2, 0, 4) + t(2, 4, -2)$$
  $$= (2t - 2, 4t, 4 - 2t).$$
Example

Find the parametric equations of the line \( \ell \) through \((-2, 0, 4)\) in the direction
\[
v = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}
\]
\[
= (2, 4, -2).
\]

- Define \( \mathbf{p} = P_0 = (-2, 0, 4) \).
- A parametrisation of \( \ell \) is
\[
\ell: \mathbf{r}(t) = \mathbf{p} + t\mathbf{v} = (-2, 0, 4) + t(2, 4, -2)
\]
\[
= (2t - 2, 4t, 4 - 2t).
\]
- The parametric equations of \( \ell \) are
\[
x = 2t - 2, \quad y = 4t, \quad z = 4 - 2t, \quad t \in \mathbb{R}.
\]
Example

Find the parametric equations of the line $\ell$ through $P = (-3, 2, -3)$ and $Q = (1, -1, 4)$. 

\[
\begin{align*}
\text{Example 2} \\
\text{Example 2} \\
\text{Example 2}
\end{align*}
\]
Example

Find the parametric equations of the line $\ell$ through $P = (-3, 2, -3)$ and $Q = (1, -1, 4)$.

- Define $\mathbf{p} = \overrightarrow{OP} = (-3, 2, -3)$ and $\mathbf{v} = \overrightarrow{PQ} = (1, -1, 4) - (-3, 2, -3) = (4, -3, 7)$. 

Example

Find the parametric equations of the line $\ell$ through $P = (-3, 2, -3)$ and $Q = (1, -1, 4)$.

- Define $\mathbf{p} = \overrightarrow{OP} = (-3, 2, -3)$ and $\mathbf{v} = \overrightarrow{PQ} = (1, -1, 4) - (-3, 2, -3) = (4, -3, 7)$.
- A parametrisation of $\ell$ is
  
  \[ \ell: \mathbf{r}(t) = \mathbf{p} + t\mathbf{v} = (-3, 2, -3) + t(4, -3, 7) = (4t - 3, 2 - 3t, 7t - 3). \]
Example

Find the parametric equations of the line \( \ell \) through \( P = (−3, 2, −3) \) and \( Q = (1, −1, 4) \).

- Define \( \mathbf{p} = \overrightarrow{OP} = (−3, 2, −3) \) and \( \mathbf{v} = \overrightarrow{PQ} = (1, −1, 4) − (−3, 2, −3) = (4, −3, 7) \).
- A parametrisation of \( \ell \) is
  \[
  \ell: \mathbf{r}(t) = \mathbf{p} + t\mathbf{v} = (−3, 2, −3) + t(4, −3, 7)
  = (4t − 3, 2 − 3t, 7t − 3).
  \]
- The parametric equations of \( \ell \) are
  \[
  x = 4t − 3, \quad y = 2 − 3t, \quad z = 7t − 3, \quad t \in \mathbb{R}.
  \]
Summary

- A parametrisation of the line through a point $P$ parallel to a vector $v \neq 0$ is
  \[ p + tv, \quad t \in \mathbb{R}, \]
  with support vector $p = \overrightarrow{OP}$ and direction vector $v$.
- A parametrisation of the line through two points $P$ and $Q$ is
  \[ p + tv, \quad t \in \mathbb{R} \]
  with support vector $p = \overrightarrow{OP}$ and direction vector $v = \overrightarrow{PQ}$.
Summary

- A parametrisation of the line through a point \( P \) parallel to a vector \( v \neq 0 \) is
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  with support vector \( p = \overrightarrow{OP} \) and direction vector \( v = \overrightarrow{PQ} \).

Warning

Parametrisations are not unique:

- Every point on the line can be chosen as support vector.
- Every non-zero vector parallel to the line can be chosen as direction vector.
Suppose two lines $\ell$ and $m$ have parametrised vector forms $p + tv$ and $q + sw$ respectively.
Suppose two lines \( \ell \) and \( m \) have parametrised vector forms \( \mathbf{p} + tv \) and \( \mathbf{q} + sw \) respectively.

An intersection is found if there are values for \( t \) and \( s \) such that

\[
\mathbf{p} + tv = \mathbf{q} + sw.
\]

\((*)\)
Intersection of lines in $\mathbb{R}^3$

- Suppose two lines $\ell$ and $m$ have parametrised vector forms $\mathbf{p} + tv$ and $\mathbf{q} + sw$ respectively.
- An intersection is found if there are values for $t$ and $s$ such that
  \[ \mathbf{p} + tv = \mathbf{q} + sw. \] (*)
- Since vector equations in $\mathbb{R}^3$ yield *three* equations, equation (*) may fail to have a solution, even if $\ell$ and $m$ are not parallel.
Suppose two lines $\ell$ and $m$ have parametrised vector forms $\mathbf{p} + tv$ and $\mathbf{q} + sw$ respectively.

An intersection is found if there are values for $t$ and $s$ such that

$$\mathbf{p} + tv = \mathbf{q} + sw.$$  \hfill (\ast)

Since vector equations in $\mathbb{R}^3$ yield three equations, equation (\ast) may fail to have a solution, even if $\ell$ and $m$ are not parallel.

Non-parallel lines that do not intersect are called skew.
Example

Let \( \ell \) be the line with support vector \((-3, -3, 1)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.
Example

Let $\ell$ be the line with support vector $(-3, -3, 1)$ and direction vector $(2, 1, 1)$. Let $m$ be the line with support vector $(2, -3, -2)$ and direction vector $(-1, 2, 4)$. Determine if $\ell$ and $m$ intersect, and if so, find the intersection point.

- Solve $s$ and $t$ from

  \[-3 + 2t = 2 - s \]
  \[-3 + t = -3 + 2s \]
  \[1 + t = -2 + 4s \]
Example

Let $\ell$ be the line with support vector $(-3, -3, 1)$ and direction vector $(2, 1, 1)$. Let $m$ be the line with support vector $(2, -3, -2)$ and direction vector $(-1, 2, 4)$. Determine if $\ell$ and $m$ intersect, and if so, find the intersection point.

- Solve $s$ and $t$ from

\[
-3 + 2t = 2 - s \\
-3 + t = -3 + 2s \\
1 + t = -2 + 4s
\]

- From the first equation follows: $s = 5 - 2t$. 

Lines $\ell$ and $m$ do no intersect.
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Let $\ell$ be the line with support vector $(-3, -3, 1)$ and direction vector $(2, 1, 1)$. Let $m$ be the line with support vector $(2, -3, -2)$ and direction vector $(-1, 2, 4)$. Determine if $\ell$ and $m$ intersect, and if so, find the intersection point.

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  \begin{align*}
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  From this follows: $t = 2$. 

Let \( \ell \) be the line with support vector \((-3, -3, 1)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.

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  - This implies \( s = 5 - 2 \cdot 2 = 1.\)
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Let \( \ell \) be the line with support vector \((-3, -3, 1)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.

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- Now check the last equation: \( 1 + t = 3 \) and \( -2 + 4s = 2 \): the equation does not hold.

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- Lines \( \ell \) and \( m \) do no intersect.
Example

Let \( \ell \) be the line with support vector \((-3, -3, 0)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.
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    The intersection point is $(1, -1, 2)$.
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- The intersection point is \((1, -1, 2)\).
Exercises

- Section 12.5: 1, 3, 5, 61.
Planes in space

**Definition**

A plane in $\mathbb{R}^3$ is defined by an equation of the form

$$M : ax + by + cz = d$$

with $a, b, c$ and $d$ real numbers.

**Examples:**
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with \( a, b, c \) and \( d \) real numbers.

**Examples:**

- The plane \( M_1 \) defined by
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  passes through the points \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\).
Planes in space

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- The plane $M_1$ defined by
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- The plane $M_2$ defined by
  $$M_2 : x + y + z = 0$$
  passes through $O$ and is parallel to $M_1$.

- The plane $M_3$ defined by
  $$M_3 : 2y = 3$$
  is the plane through $(0, 3/2, 0)$ parallel to the $xz$-plane.
Definition

A support vector of a plane $M$ is a vector $p = \overrightarrow{OP}$ with $P$ a point of $M$. 

Suppose $M$ is defined by $ax + by + cz = d$, and let $P = (x_0, y_0, z_0)$ be a point in $M$, then $ax_0 + by_0 + cz_0 = d$, hence $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ for all $(x, y, z)$ in $M$. 

Definition

The equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ is called the vector equation of $M$. 


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The equation

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**Definition**

A **normal vector** of a plane $M$ is a vector $\mathbf{n} \neq 0$ that is perpendicular to $M$.
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A normal vector of a plane $M$ is a vector $\mathbf{n} \neq 0$ that is perpendicular to $M$.

Let $M$ be a plane defined by the vector equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

then

$$\mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

for all $(x, y, z)$ in $M$. 

---

**Definition**

The equation $\mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$ is called the normal equation of $M$. 

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Introduction to Mathematics and Modeling

Lecture 6: Points, lines and planes
Definition

A normal vector of a plane $M$ is a vector $\mathbf{n} \neq \mathbf{0}$ that is perpendicular to $M$.

Let $M$ be a plane defined by the vector equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

then

$$\mathbf{n} \cdot (x - x_0, y - y_0, z - z_0)$$

$$= \mathbf{n} \cdot ((x, y, z) - (x_0, y_0, z_0))$$
Normal vectors

Definition

A normal vector of a plane $M$ is a vector $\mathbf{n} \neq \mathbf{0}$ that is perpendicular to $M$.

Let $M$ be a plane defined by the vector equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

then

$$ \begin{align*}
(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) & = (a, b, c) \cdot ((x, y, z) - (x_0, y_0, z_0)) \\
& = 0 \quad \text{for all } (x, y, z) \text{ in } M.
\end{align*} $$
Definition

A normal vector of a plane $M$ is a vector $\mathbf{n} \neq \mathbf{0}$ that is perpendicular to $M$.

- Let $M$ be a plane defined by the vector equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, then
  $$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0)$$
  $$= (a, b, c) \cdot ((x, y, z) - (x_0, y_0, z_0))$$
  $$= 0 \quad \text{for all} \ (x, y, z) \ \text{in} \ M.$$

- Define $\mathbf{x} = (x, y, z), \ \mathbf{p} = (x_0, y_0, z_0) \ \text{and} \ \mathbf{n} = (a, b, c)$, then
  $$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{for all} \ \mathbf{x} \ \text{in} \ M.$$
Definition

A normal vector of a plane $M$ is a vector $\mathbf{n} \neq \mathbf{0}$ that is perpendicular to $M$.

- Let $M$ be a plane defined by the vector equation
  \[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \]
  then
  \[
  (a, b, c) \cdot (x - x_0, y - y_0, z - z_0) \\
  = (a, b, c) \cdot ((x, y, z) - (x_0, y_0, z_0)) \\
  = 0 \quad \text{for all } (x, y, z, ) \text{ in } M.
  \]

- Define $\mathbf{x} = (x, y, z)$, $\mathbf{p} = (x_0, y_0, z_0)$ and $\mathbf{n} = (a, b, c)$, then
  \[
  n \cdot (x - p) = 0 \quad \text{for all } \mathbf{x} \text{ in } M.
  \]

Definition

The equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ is called the normal equation of $M$. 
The normal equation

**Theorem**

Let $M$ be defined by the normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, where $\mathbf{n}$ is a normal vector of $M$, and let $\mathbf{p} = (x_0, y_0, z_0)$ be a support vector. If $\mathbf{X} = (x, y, z)$ is a point of $M$ then $\mathbf{n} \perp \overrightarrow{PX}$.

- Note that $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$. 
Example

Find an equation of the plane $M$ through $(-3, 0, 7)$ orthogonal to $n = (5, 2, -1)$.
Example

Find an equation of the plane $M$ through $(-3, 0, 7)$ orthogonal to $n = (5, 2, -1)$.

- Define $p = (-3, 0, 7)$, then the normal equation $n \cdot (x - p) = 0$ gives:

$$\begin{align*}
(5, 2, -1) \cdot ((x, y, z) - (-3, 0, 7)) &= 0, \\
\text{or} \\
(5, 2, -1) \cdot (x - (-3), y - 0, z - 7) &= 0.
\end{align*}$$
The normal equation

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Find an equation of the plane $M$ through $(-3, 0, 7)$ orthogonal to $n = (5, 2, -1)$.

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- The vector equation of $M$ is

  \[
  5(x + 3) + 2y - (z - 7) = 0.
  \]
The normal equation

**Example**

*Find an equation of the plane $M$ through $(-3, 0, 7)$ orthogonal to $\mathbf{n} = (5, 2, -1)$.*

- Define $\mathbf{p} = (-3, 0, 7)$, then the normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ gives:
  
  $(5, 2, -1) \cdot ((x, y, z) - (-3, 0, 7)) = 0,$
  
  or
  
  $(5, 2, -1) \cdot (x + 3, y - 0, z - 7) = 0.$

- The vector equation of $M$ is
  
  $5(x + 3) + 2y - (z - 7) = 0.$

- Simplification gives
  
  $5x + 2y - z = -22.$
Example

Find a normal equation for the plane $M: y - 2z = 4$. 

Write the equation as follows:

$$0 \cdot x + 1 \cdot y + (-2) \cdot z = 4.$$ 

A normal vector is $n = (0, 1, -2)$. 

NB The components of $n$ are the coefficients of the equation.

For a point $P$ in the plane we choose $x = z = 0$. Then $y = 4$, so $P = (0, 4, 0)$ gives support vector $OP = p = (0, 4, 0)$.

A normal equation of $M$ is $$(0, 1, -2) \cdot (x - (0, 4, 0)) = 0.$$ 

NB Every point of $M$ can be used as support vector, for instance $p' = (1, 6, 1)$ also works.
Example

Find a normal equation for the plane $M : y - 2z = 4$.

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The normal equation

Example

Find a normal equation for the plane $M : y - 2z = 4$.

- Write the equation as follows:
  
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Find a normal equation for the plane $M : y - 2z = 4$.

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Example

Find an equation for the plane \( M \) through the points \( A = (0, 0, 1) \), \( B = (2, 0, 0) \) and \( C = (0, 3, 0) \).
Example 7

Find an equation for the plane $M$ through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.

- Suppose the equation is $ax + by + cz = d$ with yet to determine constants $a$, $b$, $c$ and $d$. 

- Substitute this in the equation:

  
  $$
  \frac{1}{2}dx + \frac{1}{3}dy + dz = d.
  $$

- Divide left- and right-hand side by $d$:

  
  $$
  \frac{1}{2}x + \frac{1}{3}y + z = 1.
  $$

- Avoid fractions by multiplying with $6$:

  
  $$
  3x + 2y + 6z = 6.
  $$

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**Introduction to Mathematics and Modeling**

**Lecture 6: Points, lines and planes**
Example

Find an equation for the plane \( M \) through the points \( A = (0, 0, 1) \), \( B = (2, 0, 0) \) and \( C = (0, 3, 0) \).

- Suppose the equation is \( ax + by + cz = d \) with yet to determine constants \( a, b, c \) and \( d \).
- The points \( A, B \) and \( C \) all lie in the plane, this gives three equations:
  \[
  \begin{align*}
  A \in M & \Rightarrow \quad c = d \quad \Rightarrow \quad c = d \\
  B \in M & \Rightarrow \quad 2a = d \quad \Rightarrow \quad a = \frac{1}{2}d \\
  C \in M & \Rightarrow \quad 3b = d \quad \Rightarrow \quad b = \frac{1}{3}d
  \end{align*}
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  \begin{align*}
  A \in M & \implies c = d \implies c = d \\
  B \in M & \implies 2a = d \implies a = \frac{1}{2}d \\
  C \in M & \implies 3b = d \implies b = \frac{1}{3}d
  \end{align*}
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- Substitute this in the equation: \( \frac{1}{2}dx + \frac{1}{3}dy + dz = d \).
Example 7

Find an equation for the plane $M$ through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.

Suppose the equation is $ax + by + cz = d$ with yet to determine constants $a$, $b$, $c$ and $d$.

The points $A$, $B$ and $C$ all lie in the plane, this gives three equations:

- $A \in M \Rightarrow c = d \Rightarrow c = d$
- $B \in M \Rightarrow 2a = d \Rightarrow a = \frac{1}{2}d$
- $C \in M \Rightarrow 3b = d \Rightarrow b = \frac{1}{3}d$

Substitute this in the equation: $\frac{1}{2}dx + \frac{1}{3}dy + dz = d$.

Divide left- and right-hand side by $d$: $\frac{1}{2}x + \frac{1}{3}y + z = 1$. 
A plane through three points

Example

Find an equation for the plane $M$ through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.

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- The points $A$, $B$ and $C$ all lie in the plane, this gives three equations:
  \[
  A \in M \implies c = d \implies c = d
  \]
  \[
  B \in M \implies 2a = d \implies a = \frac{1}{2} d
  \]
  \[
  C \in M \implies 3b = d \implies b = \frac{1}{3} d
  \]
- Substitute this in the equation: $\frac{1}{2} dx + \frac{1}{3} dy + dz = d$.
- Divide left- and right-hand side by $d$: $\frac{1}{2} x + \frac{1}{3} y + z = 1$.
- Avoid fractions by multiplying with 6: $3x + 2y + 6z = 6$.
Exercises

Section 12.5: 21, 23, 25, 27, 29.
Unlike lines in $\mathbb{R}^2$, lines in $\mathbb{R}^3$ cannot be described by one equation: a linear equation $ax + by + cz = d$ describes a plane.
Unlike lines in $\mathbb{R}^2$, lines in $\mathbb{R}^3$ cannot be described by one equation: a linear equation $ax + by + cz = d$ describes a plane.

In order to describe a line you need two equations:

$$\begin{cases} 
ax + by + cz = d \\
px + qy + rz = s
\end{cases}$$
Unlike lines in $\mathbb{R}^2$, lines in $\mathbb{R}^3$ cannot be described by one equation: a linear equation $ax + by + cz = d$ describes a plane.

In order to describe a line you need two equations:

\[
\begin{align*}
ax + by + cz &= d \\
px + qy + rz &= s
\end{align*}
\]

Regard a line as the intersection of two planes:

\[
\begin{align*}
ax + by + cz &= d \\
px + qy + rz &= s
\end{align*}
\]
Example

Give a parametrisation of the line described by the equations

\[
\begin{align*}
x + y - 2z &= -1 \\
2x - y + z &= 2
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Choose one of the variables as parameter, for instance: \( x = t \)
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- Choose one of the variables as parameter, for instance: \( x = t \)
- Replace \( x \) by \( t \) in the given equations:

\[
\begin{align*}
  y - 2z &= -1 - t \\
  -y + z &= 2 - 2t
\end{align*}
\]
Example

Give a parametrisation of the line described by the equations

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    x + y - 2z &= -1 \\
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(1)

- Express \(y\) and \(z\) in \(t\) by solving system (1). For instance, from the first equation follows

\[
y = 2z - 1 - t.
\]
Example

Give a parametrisation of the line described by the equations

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\begin{align*}
  x + y - 2z &= -1 \\
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- Choose one of the variables as parameter, for instance: \( x = t \)
- Replace \( x \) by \( t \) in the given equations:
  
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  \begin{align*}
    y - 2z &= -1 - t \\
    -y + z &= 2 - 2t
  \end{align*}
  \]
  
  \( (1) \)

- Express \( y \) and \( z \) in \( t \) by solving system (1). For instance, from the first equation follows
  
  \[
  y = 2z - 1 - t
  \]
  
  \( (2) \)

- Plug this in the second equation of (1) and solve \( z \):
  
  \[
  -(2z - 1 - t) + z = 2 - 2t \quad \Rightarrow \quad z = -1 + 3t
  \]
Example (continued)

- Use equation (2) to express $y$ in $t$:

$$y = 2z - 1 - t = 2(-1 + 3t) - 1 - t = -3 + 5t$$
Use equation (2) to express $y$ in $t$:

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Summary: the boxed equations express $x, y$ and $z$ in $t$ and therefore can be used as parametric equations for the line:

$$\begin{cases} x = t \\ y = -3 + 5t \\ z = -1 + 3t \end{cases}$$
Example (continued)

- Use equation (2) to express \( y \) in \( t \):

\[
y = 2z - 1 - t = 2(-1 + 3t) - 1 - t = -3 + 5t
\]

- Summary: the boxed equations express \( x, y \) and \( z \) in \( t \) and therefore can be used as parametric equations for the line:

\[
\begin{align*}
x &= t \\
y &= -3 + 5t \\
z &= -1 + 3t
\end{align*}
\]

- The parametrised vector form then is

\[
\mathbf{x} = (x, y, z) = (0, -3, -1) + t(1, 5, 3).
\]
Example (continued)

- Use equation (2) to express $y$ in $t$:

$$y = 2z - 1 - t = 2(-1 + 3t) - 1 - t = -3 + 5t$$

- Summary: the boxed equations express $x$, $y$ and $z$ in $t$ and therefore can be used as parametric equations for the line:

$$\begin{cases}
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- The parametrised vector form then is

$$\mathbf{x} = (x, y, z) = (0, -3, -1) + t(1, 5, 3).$$

- A support vector then is $(0, -3, -1)$, and as direction vector you can use $(1, 5, 3)$. 
Check that
\[
\begin{align*}
x + y - 2z &= -1 \\
2x - y + z &= 2
\end{align*}
\]
is the line through \( p = (0, -3, 1) \) and in direction \( v = (1, 5, 3) \).
Check your answer!

- Check that
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  x + y - 2z &= -1 \\
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  \end{align*}
  \]
  is the line through \( p = (0, -3, 1) \) and in direction \( v = (1, 5, 3) \).

- Let \( x = 0, \ y = -3, \ z = -1 \), then
  \[
  \begin{align*}
  x + y - 2z &= 0 - 3 + 2 = -1 \\
  2x - y + z &= 0 + 3 - 1 = 2
  \end{align*}
  \]
Check your answer!

- Check that
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  x + y - 2z &= -1 \\
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  x + y - 2z &= 0 - 3 + 2 = -1 \\
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  \end{align*}
  \]

- The normal vectors of the planes \( x + y - 2z = -1 \) and \( 2x - y + z = 2 \) are \( n_1 = (1, 1, -2) \) and \( n_2 = (2, -1, 1) \) respectively.
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- Check that \( \mathbf{v} \perp \mathbf{n}_1 \) and \( \mathbf{v} \perp \mathbf{n}_2 \):
  \[
  \mathbf{v} \cdot \mathbf{n}_1 = 1 + 5 - 6 = 0,
  \]
  and
  \[
  \mathbf{v} \cdot \mathbf{n}_2 = 2 - 5 + 3 = 0.
  \]
Example 10

The line \( \ell \) is defined by the parametrisation

\[
x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t, \quad t \in \mathbb{R}.
\]

Find the intersection of \( \ell \) and the plane \( 3x + 2y + 6z = 6 \).
**Example**

*The line \( \ell \) is defined by the parametrisation*

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x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t, \quad t \in \mathbb{R}.
\]

*Find the intersection of \( \ell \) and the plane \( 3x + 2y + 6z = 6 \).*

- Suppose the intersection is
  \[
  x_0 = \left( \frac{8}{3} + 2t, -2t, 1 + t \right).
  \] (1)
**Example**

The line $\ell$ is defined by the parametrisation

$$
x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t, \quad t \in \mathbb{R}.
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Find the intersection of $\ell$ and the plane $3x + 2y + 6z = 6$.

- Suppose the intersection is
  $$
x_0 = \left(\frac{8}{3} + 2t, -2t, 1 + t\right).
  \tag{1}
$$
- The point $x_0$ lies on the plane, so
  $$
  3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6.
  $$
**Example 10**

The line \( \ell \) is defined by the parametrisation

\[
\begin{align*}
    x &= \frac{8}{3} + 2t, \\
    y &= -2t, \\
    z &= 1 + t,
\end{align*}
\]

\( t \in \mathbb{R} \).

Find the intersection of \( \ell \) and the plane \( 3x + 2y + 6z = 6 \).

- Suppose the intersection is
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  \mathbf{x}_0 = \left( \frac{8}{3} + 2t, -2t, 1 + t \right).
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- The point \( \mathbf{x}_0 \) lies on the plane, so
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  3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6.
  \]

- Solve \( t \) from this equation:
  \[
  8 + 6t - 4t + 6 + 6t = 6,
  \]

which implies \( t = -1 \).
**Example**

*The line $\ell$ is defined by the parametrisation*

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x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t, \quad t \in \mathbb{R}.
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- Suppose the intersection is 
  \[x_0 = \left(\frac{8}{3} + 2t, -2t, 1 + t\right).\] (1)
- The point $x_0$ lies on the plane, so
  \[3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6.\]
- Solve $t$ from this equation:
  \[8 + 6t - 4t + 6 + 6t = 6,\]
  which implies $t = -1$.
- The intersection is obtained by substituting $t = -1$ in (1):
  \[x_0 = \left(\frac{2}{3}, 2, 0\right).\]
If a plane $M$ is defined by the equation $ax + by + cz = d$, then $\mathbf{n} = (a, b, c)$ is a normal of $M$. This is also a direction vector of the line.
If a plane $M$ is defined by the equation $ax + by + cz = d$, then $n = (a, b, c)$ is a normal of $M$. This is also a direction vector of the line.

If the line passes through $q$ then the line can be parametrised by

$$\ell: q + tn.$$
Line through a point perpendicular to a plane

- If a plane $M$ is defined by the equation $ax + by + cz = d$, then $\mathbf{n} = (a, b, c)$ is a normal of $M$. This is also a direction vector of the line.

- If the line passes through $\mathbf{q}$ then the line can be parametrised by

$$\ell: \mathbf{q} + t\mathbf{n}.$$ 

- The projection of $\mathbf{q}$ on $M$ is $\hat{\mathbf{q}}$, the intersection of the line $\ell$ with $M$. 
Example

Let \( M \) be defined by \( 2x - y - z = 3 \), and let \( q = (-2, 2, 3) \). Find the projection of \( q \) on \( M \).
Example

Let $M$ be defined by $2x - y - z = 3$, and let $q = (-2, 2, 3)$. Find the projection of $q$ on $M$.

- The coefficients of the equation for $M$ give the normal:
  
  $n = (2, -1, -1)$. 

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Let $M$ be defined by $2x - y - z = 3$, and let $q = (-2, 2, 3)$. Find the projection of $q$ on $M$.

- The coefficients of the equation for $M$ give the normal:
  \[ n = (2, -1, -1). \]
- The line through $p$ perpendicular to $M$ is parametrized by
  \[ q + tn = (-2, 2, 3) + t(2, -1, -1) = (2t - 2, -t + 2, -t + 3). \]
Example

Let $M$ be defined by $2x - y - z = 3$, and let $q = (-2, 2, 3)$. Find the projection of $q$ on $M$.

- The coefficients of the equation for $M$ give the normal:
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- The parametric equations are
  $$x = 2t - 2, \quad y = -t + 2 \quad \text{and} \quad z = -t + 3.$$
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- Substitution in the equation for $M$ give the equation
  \[ 2(2t - 2) - (-t + 2) - (-t + 3) = 3 \quad \Rightarrow \quad 6t - 9 = 3. \]
Example

Let $M$ be defined by $2x - y - z = 3$, and let $q = (-2, 2, 3)$. Find the projection of $q$ on $M$.

- The coefficients of the equation for $M$ give the normal: $n = (2, -1, -1)$.
- The line through $p$ perpendicular to $M$ is parametrized by $q + tn = (-2, 2, 3) + t(2, -1, -1) = (2t - 2, -t + 2, -t + 3)$.
- The parametric equations are $x = 2t - 2$, $y = -t + 2$ and $z = -t + 3$.
- Substitution in the equation for $M$ gives the equation $2(2t - 2) - (-t + 2) - (-t + 3) = 3$ $\Rightarrow$ $6t - 9 = 3$.
- Solving this equation yields $t = 2$. 
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  \[ 2(2t - 2) - (-t + 2) - (-t + 3) = 3 \quad \Rightarrow \quad 6t - 9 = 3. \]
- Solving this equation yields $t = 2$.
- Substitution of $t = 2$ in the parametric equations gives the intersection:
  \[ \hat{q} = (2, 0, 1) \]
Distance of a point to a plane

**Theorem**

If \( P \) is a point on the plane \( M \), and \( \mathbf{n} \) is a normal of \( M \), the distance of an arbitrary point \( Q \) to \( M \) is

\[
d = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|}
\]
**Theorem**

If $P$ is a point on the plane $M$, and $n$ is a normal of $M$, the distance of an arbitrary point $Q$ to $M$ is

$$d = \frac{|PQ \cdot n|}{|n|}$$

- The distance can be found by calculating $|q - \hat{q}|$, where $q = \overrightarrow{OQ}$ and $\hat{q}$ is the projection of $q$ on $M$. 

Section 12.5, page 711
The plane $M$ is defined by $3x + 2y + 6z = 6$. Find the distance of $Q = (1, 1, 3)$ to $M$. 
Example

The plane $M$ is defined by $3x + 2y + 6z = 6$. Find the distance of $Q = (1, 1, 3)$ to $M$.

- The coefficients of the equation for $M$ give the normal:
  \[ \mathbf{n} = (3, 2, 6). \]
The plane $M$ is defined by $3x + 2y + 6z = 6$. Find the distance of $Q = (1, 1, 3)$ to $M$.

- The coefficients of the equation for $M$ give the normal:
  \[ \mathbf{n} = (3, 2, 6). \]

- For a point of $M$, choose two values for say $x$ and $z$, then $y$ follows from the equation:
  \[ x = 0, \ z = 0 \quad \Rightarrow \quad y = 3, \]
  hence $P = (0, 3, 0)$ is a point of $M$. 

\[ \text{The distance of } Q \text{ to } M \text{ is} \]
\[ d = \left| \frac{(1 - 0, 1 - 3, 3 - 0) \cdot (3, 2, 6)}{|n|} \right| = \frac{|(-2, -2, 3) \cdot (3, 2, 6)|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{17}{7}. \]
Example

The plane $M$ is defined by $3x + 2y + 6z = 6$. Find the distance of $Q = (1, 1, 3)$ to $M$.

- The coefficients of the equation for $M$ give the normal:
  $$n = (3, 2, 6).$$
- For a point of $M$, choose two values for say $x$ and $z$, then $y$ follows from the equation:
  $$x = 0, \ z = 0 \implies y = 3,$$
  hence $P = (0, 3, 0)$ is a point of $M$.
- The distance of $Q$ to $M$ is
  $$d = \frac{|PQ \cdot n|}{|n|} = \frac{|((1, 1, 3) - (0, 3, 0)) \cdot (3, 2, 6)|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{17}{7}.$$
Parametrise the line $\ell$ as follows:

$$\ell : \mathbf{r}(t) = (x_0, 0, 0) + t(x_1 - x_0, y_1, z_1), \quad t \in \mathbb{R}.$$
- Parametrise the line \( \ell \) as follows:
\[
\ell : \mathbf{r}(t) = (x_0, 0, 0) + t(x_1 - x_0, y_1, z_1), \quad t \in \mathbb{R}.
\]
- The intersection of \( \ell \) and the \( yz \)-plane is \( P = \mathbf{r}(t_0) \) with \( t_0 = \frac{x_0}{x_0 - x_1} \).
Parametrise the line \( \ell \) as follows:
\[
\ell : \mathbf{r}(t) = (x_0, 0, 0) + t(x_1 - x_0, y_1, z_1), \quad t \in \mathbb{R}.
\]

The intersection of \( \ell \) and the \( yz \)-plane is \( P = \mathbf{r}(t_0) \) with 
\[
t_0 = \frac{x_0}{x_0 - x_1}.
\]

For \( P = (0, y, z) \) we have
\[
y = t_0 y_1 = \frac{x_0 y_1}{x_0 - x_1} \quad \text{and} \quad z = t_0 z_1 = \frac{x_0 z_1}{x_0 - x_1}.
\]
Exercises

Section 12.5: 39, 45, 53, 57, 74.